Ergodicity Breaking in a Deterministic Dynamical System

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Abstract

The concept of weak ergodicity breaking is defined and studied in the context of deterministic dynamics. We show that weak ergodicity breaking describes a weakly chaotic dynamical system: a nonlinear map which generates subdiffusion deterministically. In the non-ergodic phase non-trivial distribution of the fraction of occupation times is obtained. The visitation fraction remains uniform even in the non-ergodic phase. In this sense the non-ergodicity is quantified, leading to a statistical mechanical description of the system even though it is not ergodic.

INTRODUCTION

The relation between deterministic non-linear dynamics and statistical mechanics is a topic of wide interest [1, 2]. In particular, the domain of validity of the ergodic hypothesis for low dimensional dynamical systems is fundamental in our understanding of the foundations of statistical mechanics. In usual setting of statistical mechanics the phase space of a closed system is divided into equally sized cells, then if the system is ergodic it will spend equal amount of time in each cell in the limit of long measurement time. Ofcourse many dynamical systems are non-ergodic, for example the phase space may decompose into regions, where the system cannot access one part of the phase space if it started in a different part. Such non-ergodicity is called here strong non-ergodicity.

A second type of non-ergodicity, called weak non-ergodicity is investigated in this manuscript. As conjectured by Bouchaud [3] in the context of glassy dynamics, in weak ergodicity breaking the phase space is *a priori* not broken into mutually inaccessible regions. However the system does not spend equal amount of time in equally sized regions of phase space (see detailed definition below). While occupation times of cells in phase space remain random, we showed in the context of *stochastic* dynamics that distribution of occupation times is universal [4] in a way that will be made clearer later. The goal of this paper is to investigate weak non-ergodicity for a *deterministic* process.

In terms of stochastic theories like the trap model, the random energy model [3] and the continuous time random walk [4] it is known that when the dynamics exhibits aging and anomalous diffusion it may exhibit weak ergodicity breaking. It is also well known that a wide variety of deterministic dynamical systems exhibit anomalous diffusion [5, 6, 7, 8]. Thus the question arises: Can deterministic dynamical systems exhibit weak ergodicity breaking? In this manuscript a detailed definition of weak ergodicity breaking (WEB) in the context of deterministic dynamics is introduced. It is shown that that a low dimensional model, a deterministic non linear map which has no disorder built into it, yields WEB. Non-trivial (in a way defined later) probability density function (PDF) of fraction of occupation time is shown to describe equilibrium properties of the dynamical system. In this sense the non-ergodicity is quantified, enabling us to perform statistical mechanical description of the system although it is not ergodic.

GEISEL'S MAPS

Probably the simplest deterministic maps which lead to normal and anomalous diffusion are one dimensional maps of the form

$$x_{t+1} = x_t + F(x_t), \tag{1}$$

with the following symmetry properties of $F(x)$: (i) $F(x)$ is periodic with a period interval set to 1, i.e., $F(x) = F(x+N)$, where N is an integer. (ii) $F(x)$ is anti-symmetric; namely, $F(x) = -F(-x)$, note that t in Eq. (1) is a discrete time. Geisel and Thomae [9] considered a wide family of such maps which behave like

$$F(x) = ax^z \quad \text{for} \quad x \to 0_+ \tag{2}$$

where $z > 1$. Eq. (2) defines the property of the map close to its unstable fixed point. In numerical experiments introduced later we will use the following map

$$F(x) = (2x)^z, \qquad 0 \le x \le 1/2 \tag{3}$$

which together with the symmetry properties of the map define the mapping for all x. The functional form of the map on a unit interval was introduced by Pomeau and Manneville to describe intermittency [10]. Variations of these maps have been investigated by taking into account time dependent noise [11], quenched disorder [12], and additional uniform bias which breaks the symmetry of the map [13]. We consider the map with L unit cells and periodic boundary conditions. In Fig. 1 we show the map, and one realization of a path along three unit cells.

The dynamics of the map in the vicinity of the unstable fixed points, is governed by a q *exponential* sensitivity of the trajectories to initial conditions, i.e., weak chaos. The role of such q *exponential* in models related to ours (the logistic map) and Tsallis statistics is a subject of ongoing research [14, 15, 16]. Grassberger [17] claimed that ideas related to Tsallis statistics are wrong due to misconceptions on basic notions of ergodic theory. here we show that for the weakly chaotic map under consideration, the correct statistical framework is given by the concept of WEB.

In an ongoing process a walker following the iteration rules may get stuck close to the vicinity of one of the unstable fixed points of the map. It has been shown both analytically

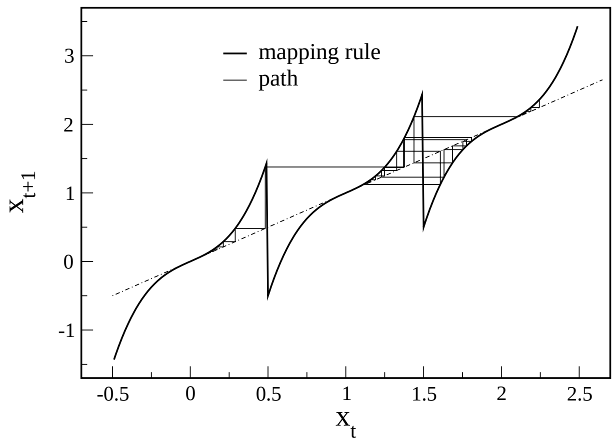

FIG. 1: The mapping Eqs. (1,3) with z=3 and L=3, the dashed line shows the curve $x_{t+1} = x_t$. Fixed points are at $x_t = 0, 1, 2$. A specific realization of the dynamics is also drawn (thin line).

and numerically, that the PDF of escape times from the vicinity of the fixed points decays as a power law [9]. To see this, one considers the dynamics in a half unit cell, say $0 < x < 1/2$. Assume that at time $t = 0$ the particle is on x^* residing in the vicinity of the fixed point $x = 0$. Close to the fixed point we may approximate the map Eq. (1) with the differential equation $dx/dt = F(x) = ax^z$. This equation is reminiscent of the equation defining the q generalized Lyaponov exponent [16]. The solution is written in terms of the q *exponential* function, $\exp_q(y) \equiv [1 + (1-q)\,y]^{1/(1-q)}$ where $q = z$ and

$$x(t) = x^* \exp_z\left[a x^{*(z-1)} t\right]. \tag{4}$$

We invert Eq. (4) to obtain the escape time from x^* to a boundary on $b(x^* < b < 1/2)$ which is $t \simeq \int_{x^*}^{b} [F(x)]^{-1} dx$, using Eq. (2) we obtain $t \simeq (1/a)\left[(x^{*1-z})/(z-1) - (b^{1-z})/(z-1)\right]$ a $\ln_q$ (i.e., the inverse function of $\exp_q$) behaviour. The PDF of escape times $\psi(t)$ is related to the unknown PDF of injection points $\eta(x^*)$, through the chain rule $\psi(t) = \eta(x^*)\,|dx^*/dt|$. Expanding $\eta(x^*)$ around the unstable fixed point $x^* = 0$ one finds that for large escape times

$$\psi(t) \sim \frac{A}{|\Gamma(-\alpha)|} t^{-1-\alpha}, \quad \alpha = (z-1)^{-1} \tag{5}$$

where A depends on the PDF of injection points, namely on how trajectories are injected from one cell to the other. The parameter A is sensitive to the detailed shape of the map while the parameter z depends only on the behaviour of the map close to the unstable fixed

points. When $z > 2$ the average escape time diverges [5, 9, 18], and in turn yields aging [19], non-stationarity [20], anomalous diffusion [21, 22] and as we show here WEB.

WEAK ERGODICITY BREAKING

The map is said to be ergodic if for almost any initial condition (excluding paths starting at fixed points) the path spends equal amounts of time in each cell in the limit of long measurement time. Namely $t^1/t = 1/L$, where t^1 is the time spent in one of the lattice cells and t is the total measurement time. Another method to quantify the dynamics is to consider the fraction of number of visits per cell n^1/n where n^1 is the number of visits in one of the lattice cells and n is the total number of visits (or total number of jumps between cells). For maps with finite average escape time $\langle\tau\rangle = \int_0^\infty \tau\psi(\tau)\,d\tau$, i.e., $z > 2$, the total measurement time $t \sim n\langle\tau\rangle$ and the occupation time of one cell $t^1 \sim n^1\langle\tau\rangle$ hence $n^1/n = 1/L$, the equalities above should be understood in statistical sense. We call n^1/n the visitation fraction, which plays an important role in WEB.

More generally the fraction of occupation times (visitation fraction) in $m \leq L$ cells is $t^m/t\,(n^m/n)$, where t^m (n^m) is the occupation time (number of visits) in m cells respectively. For ergodic systems the PDF of the fraction of occupation time is

$$f\left(\frac{t^m}{t}\right) = \delta\left(\frac{t^m}{t} - \frac{m}{L}\right), \tag{6}$$

and the visitation fraction is

$$\frac{n^m}{n} = \frac{m}{L} \tag{7}$$

in statistical sense, both equalities are valid only in the limit of long measurement time $(t \to \infty)$. Note that the specific choice of the m cells is not important. Maps exhibiting strong non-ergodicity do not obey Eqs. (6,7). Dynamics obeying Eq. (7) but not Eq. (6) are said to be weakly non-ergodic. As we will show when $z > 2$ WEB is found.

We start our analysis by discussing numerical simulations of the map with system of size $L = 9$. In Fig. 2 (a) we show two paths generated according to the mapping rule Eqs. (2,3) with z=3. As one can see each path gets stuck at one of the cells for a time which is of the order of the measurement time. During the measurement the path visits all the L cells many times. In Fig. 2 (b) we consider the case $z = 1.5$, we show that the fraction of occupation time obeys Eq. (6) with $m = 1$ namely $t^1/t = 1/L$, with $L = 9$. On the other hand in Fig. 2

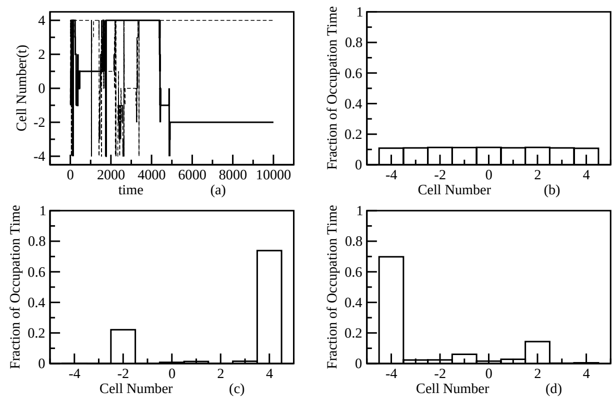

FIG. 2: (a) Two paths (dashed and solid curves) generated according to the mapping rule with z=3, for each path the particle gets stuck in one of the cells for a time which is of the order of the measurement time. (b) Histogram of the fraction of occupation for a path with z=1.5. For this ergodic case the fraction of occupation times in the cells are equal. (c)+(d) The same as in (b), but this time for the non-ergodic phase with z=3.

(c,d) we show the fraction of occupation time histogram for two initial conditions obtained using $z = 3$. Since each path is localized in one of the cells for a time which is of the order of the total measurement time, the histogram for this case looks very different than that of the ergodic case and clearly Eq. (6) is not valid. Histograms of the visitation fraction in each cell are presented in Fig. 3, for both ergodic ($z = 1.2$) (a) and non-ergodic ($z = 3$) (b) maps. The Fig. demonstrates that for both cases Eq. (7) holds, i.e., the visitation fraction in each cell is given by $1/L$. From Fig. 2 we see that the fraction of occupation time is random, hence we investigate its distribution. We will find universal PDFs of occupation time, which generalize the ergodic rule Eq. (6).

The PDF for t^m(the occupation time of m cells) in the case of diverging mean escape time is now derived. In particular we show that the assumption Eq. (7) with power law waiting times, leads to nontrivial PDFs of fraction of occupation times of the map. We denote by $f^{out}_{n,t}\left(t^m\right)$ the PDF of t^m, in the case that the path isn't within the m cells at time t and that during the measurement time t there were n jumps. Then

$$f^{out}_{n,t}\left(t^m\right) = \left\langle \delta\left(t^m - \sum_{k=1}^{n_i} \tau^m_k\right) I\left(T_n < t < T_{n+1}\right)\right\rangle, \tag{8}$$

where the summation in the delta function is over all sojourn times within the m cells, namely τ^m_k is the $k'th$ sojourn time within the m cells, T_n is the time after n jumps and $I\left(x\right)$

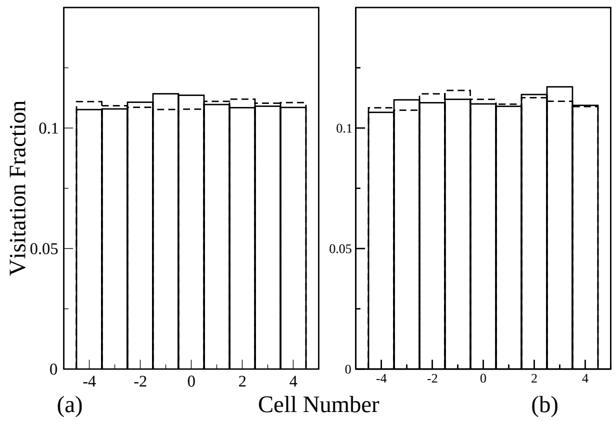

FIG. 3: Histogram of the visitation fraction in each of the 9 lattice cells. (a) z=1.2 dashed and solid lines correspond to two initial conditions. (b) The same as (a) but now z=3. In both cases the visitation fractions in each cell are equal, with small fluctuations.

is equal 1 if x is true and 0 otherwise, the brackets denote average over all sojourn times τ. Double Laplace transform (LT) of Eq. (8) yields $\widetilde{f}^{out}_{n,s}(u) =$

$$\left\langle \int_0^\infty \int_0^\infty e^{-ut^m} e^{-st} \delta\left(t_i - \sum_{k=1}^{n^m} \tau_k^m\right) I\left(T_n < t < T_{n+1}\right) dt_i dt \right\rangle$$

$$= \frac{\widetilde{\psi}^{n^m}(u+s)\,\widetilde{\psi}^{n-n^m}(s)\left[1-\widetilde{\psi}(s)\right]}{s}, \tag{9}$$

where $\widetilde{\psi}(s)$ is the LT of $\psi(t)$. We assumed that waiting times in cells are not correlated. This and other assumptions are checked later using numerical simulations. In a way similar to Eq. (9) we denote by $f^{in}_{n,t}(t^m)$ the PDF of t^m in the case that the particle is within the m cells at the end of the measurement, then in double Laplace space we find

$$\widetilde{f}^{in}_{n,s}(u) = \widetilde{\psi}^{n^m}(u+s)\,\widetilde{\psi}^{n-n^m}(s)\,\frac{1-\widetilde{\psi}^{n^m+1}(u+s)}{u+s}. \tag{10}$$

Considering an ensemble of initial conditions uniformly distributed in a cell, the probability for the path to be within the m cells at the end of the measurement is m/L, thus the double Laplace transformed PDF of the occupation time of the cell, given that there were n visits during the measurement is

$$\widetilde{f}_{n,s}(u) = \frac{m}{L}\widetilde{f}^{in}_{n,s}(u) + \left(1-\frac{m}{L}\right)\widetilde{f}^{out}_{n,s}(u). \tag{11}$$

Using Eqs. (9,10) and the assumption $n^m/n = m/L$ Eq. (7)we rewrite $\widetilde{f}_{n,s}(u)$ as

$$\widetilde{f}_{n,s}(u) = \left[\frac{m}{L}\frac{1-\widetilde{\psi}(u+s)}{s+u} + \frac{L-m}{L}\frac{1-\widetilde{\psi}(s)}{s}\right] \left(\widetilde{\psi}^{m/L}(u+s)\,\widetilde{\psi}^{\frac{L-m}{L}}(s)\right)^n . \tag{12}$$

Let $f_t(t^m)$ be the PDF of occupation time of the m cells, its double LT is given by

$$\begin{aligned}\widetilde{f}_s(u) &= \sum_{n=0}^{\infty}\widetilde{f}_{n,s}(u) \\ &= \left[\frac{m}{L}\frac{1-\widetilde{\psi}(u+s)}{s+u} + \frac{L-m}{L}\frac{1-\widetilde{\psi}(s)}{s}\right] \frac{1}{1-\widetilde{\psi}^{m/L}(u+s)\,\widetilde{\psi}^{\frac{L-m}{L}}(s)}.\end{aligned} \tag{13}$$

Taking the limit $u, s \to 0$ (using the asymptotic behaviour of the escape time PDF $\widetilde{\psi}(s) \sim 1 - As^{\frac{1}{z-1}}$), and inverting the double LT one obtains Lamperti PDF [23] of fraction of occupation time

$$f\left(\frac{t^m}{t}\right) = \frac{\sin\left(\frac{\pi}{z-1}\right)}{\pi} \frac{R\left(\frac{t^m}{t}\right)^{\frac{2-z}{z-1}}\left(1-\frac{t^m}{t}\right)^{\frac{2-z}{z-1}}}{R^2\left(1-\frac{t^m}{t}\right)^{\frac{2}{z-1}} + \left(\frac{t^m}{t}\right)^{\frac{2}{z-1}} + 2R\left(1-\frac{t^m}{t}\right)^{\frac{1}{z-1}}\left(\frac{t^m}{t}\right)^{\frac{1}{z-1}}\cos\left(\frac{\pi}{z-1}\right)} \tag{14}$$

where $R = \frac{m}{L-m}$, which is valid only when t is large. Eq. (14) for $m = 1$ is in agreement with the formalism developed in [4]. Here we showed that the WEB condition $n^m/n = m/L$ (i.e., the visitation fraction is equal to the probability of occupying the m cells in *ensemble* sense) and the power law behaviour of the sojourn time PDF lead to the statistical law Eq. (14).

The assumptions made in the derivation of Eq. (14) are tested using numerical simulations of the map. Plots of the PDF of fraction of occupation time for three different values of z are shown in Fig. 4. We used 10^6 trajectories and measured the occupation time of $m = 3$ cells in a system of size $L = 9$. In the ergodic case $z < 2$ the PDF is just a delta function centered at m/L (see Fig. 4 (a)). In the non-ergodic cases the maxima of $f(t^m/t)$ are at 0 and 1. These events $t^m/t \approx 1(t^m/t \approx 0)$ correspond to trajectories where the particle occupies (does not occupy) one of the observed cells for the whole duration of the measurement, respectively. In the non ergodic phase $z > 2$ two types of behaviors are found. For $z = 2.25$ the PDF of fraction of occupation time has a W shape, with a peak in the vicinity of m/L. While for $z = 3$, we find a U shape PDF and the probability of obtaining

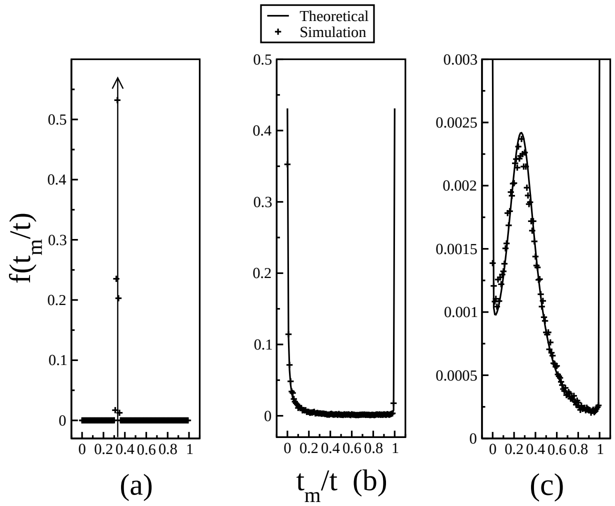

FIG. 4: (a) The PDF of the fraction of occupation time of m=3 cells out of L=9 cells for z=1.5, In this ergodic case the PDF is very narrow and centered around m/L. (b) Same as (a), however here z=3. The PDF has a U shape, with maxima at 0 and 1. (c) Same as (a) with z=2.25. In this case a W shaped PDF is obtained, with a peak at the vicinity of m/L, but yet the maxima are at 0 and 1. The curve is Eq. (14) without fitting. Only the parameter z determines the shape of the PDFs, while detailed shape of the map is unimportant.

fraction of occupation time equal to m/L is almost zero. Comparison of theory Eq. (14) with the numerical simulations of the map shows excellent agreement without any fitting, thus justifying our assumptions. Note that if we limit our model to integer z (i.e., assuming an analytical behavior) then we either observe WEB ($z > 2$) or the special border point between WEB and ergodicity $z = 2$. The latter point exhibits very slow convergence and logarithmic corrections which should be the subject of future work.

CONCLUSION

A detailed characterization of weak ergodicity breaking was established as follows: (i) The phase space is not divided into mutually inaccessible regions, thus the system visits all cells in its phase space for almost all (all but a set of measure zero) initial conditions. (ii) The visitation fraction fluctuates very slightly in the limit $t \to \infty$ (note that n and n^m exhibit strong fluctuations). (iii) The visitation fraction is determined by the probability of finding a member of an ensemble of particles within the given cell. (iv) The total time the

system spends in each cell fluctuates strongly, and is described by Eq. (14), implying that statistics of occupation times depend only on the parameter z describing the non-linearity and not on the shape of the map far from the fixed points (e.g. a in Eq. (2)). A relation between weak chaos, deterministic anomalous diffusion and weak ergodicity breaking was found. Since the former behaviors are wide spread in dynamical systems, it is possible that the non-ergodic dynamical scenario might emerge in other fractional dynamical systems, and then the present concepts naturally lead to a new type of weakly non-ergodic statistical mechanics.

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[1] J. R. Dorfman An Introduction to Chaos in Nonequilibrium Statistical Mechanics (Cambridge Lecture Notes in Physics, 1999).

[2] G. M. Zaslavsky Physics Today **8**, 39 (1999).

[3] J. P. Bouchaud J. Phys. I France **2**, 1705 (1992).

[4] G. Bel and E. Barkai, Phys. Rev. Lett **94**, 240602 (2005).

[5] G. M. Zaslavski, Phys. Rep. **371**, 461 (2002).

[6] J. Klafter, M. F. Shlesinger, and G. Zumofen, Phys. Today **49**, No. 2, 33 (1996).

[7] S. Tasaki and P. Gaspard, Physica D **187**, 51 (2004).

[8] R. Arutso and G. Cristadoro, Phys. Rev. Lett. **90**, (2003).

[9] T. Geisel and S. Thomae, Phys. Rev. Lett. **52**, 1936 (1984).

[10] Y. Pomeau and P. Manneville, Commun. Math. Phys. **74**, 189 (1980); P. Manneville, J. Phys. **41**, 1235 (1980).

[11] R. Bettin, R. Mannella, B. J. West and P. Grigolini, Phys. Rev. E **51**, 212 (1995).

[12] G. Radons, Phys. Rev. Lett. **77**, 4748 (1996).

[13] E. Barkai and J. Klafter, Phys. Rev. Lett. **79**, 2245 (1997).

[14] F. Baldovin, A. Robledo, Europhys. Lett. **60**, 518 (2002).

[15] A. Robledo, Physics A **342**, 104 (2004).

[16] G. F. J. Ananos, C. Tsallis, Phys. Rev. Lett. **93**, 020601 (2004).

[17] P. Grassberger, cond-mat/0508110 (accepted to Phys. Rev. Lett.).

[18] T. Geisel, J. Nierwetberg and A. Zacherl, Phys. Rev. Lett. **54**, 616 (1985).

[19] E. Barkai, Phys. Rev. Lett. **90**, 104101 (2003).

[20] M. Ignaccolo, P. Grigolini, and A. Rosa, Phys. Rev. E **64**, 026210 (2001).

[21] G. Zumofen and J. Klafter, Phys. Rev. E **47**, 851 (1993).

[22] N. Korabel, A. V. Chechkin, R. Klages, I. M. Sokolov and V. Yu. Gonchar, Europhys. Lett. **70**, 63 (2005).

[23] J. Lamperti, Trans. Amer. Math. Soc. **88**, 380 (1958).